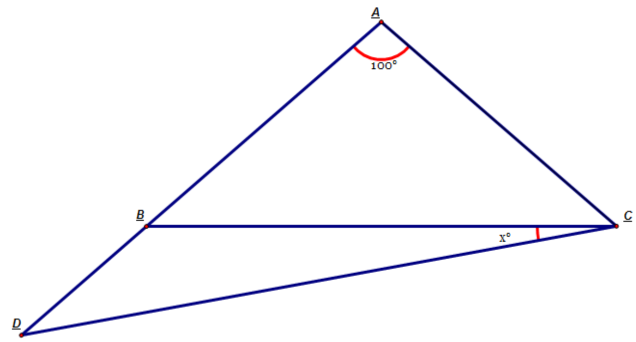
100 deg Isosceles Triangle

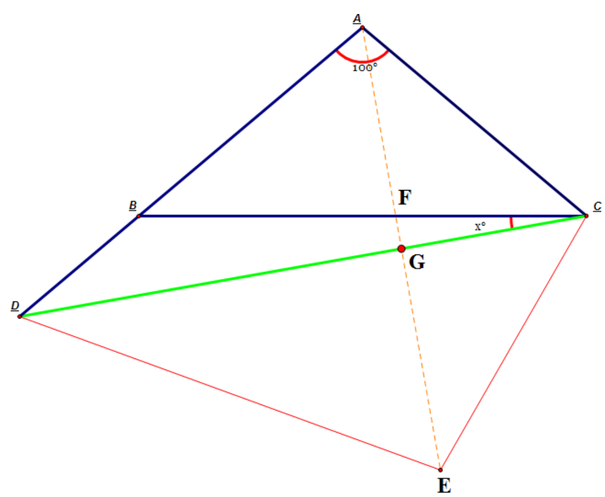
Problem: Given an isosceles triangle ABC with AB = AC and the measure of angle BAC = 100 degrees. Extend AB to point D such that AD = BC. Now draw segment CD. What is the measure of angle BCD? Prove your results by geometric reasoning, rather than measuring.



Solution:

Given $\Delta ABC$is isosceles with $AB = AC$ and $\angle BAC = 100\,°\Longrightarrow \angle ABC = \angle ACB = 40\,°$.

Now, let's look at the Equilateral Triangle, ΔADE, constructed below:



So, $\angle BAC = \angle CAF + \angle BAF = 100\,°$.

This implies that $\angle CAF = 40\,°$ (since $\angle BAF = 60°$)

So, $\angle CAF \cong \angle ACF$. Hence ΔACF is also isosceles. So, $AF = CF$.

Now, since $BC = AD = AE$. It implies, $BC = AE \Longrightarrow BF + FC = AF + FE$

We have seen that $FC = AF$, so we have $BF = FE$

Additionally, $\angle AFB = \angle CFE$ (since vertical angles)

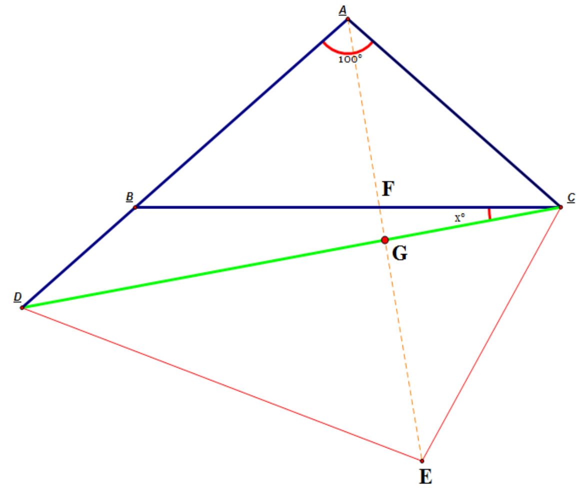
Hence, $\Delta AFB \cong CFE$

So, $AB = CE = CA$.

Now, considering ΔADC and ΔCDE we have

$AC = CE$, $AD = DE$, and CD is common

So, $\Delta ADC \cong \Delta CDE$ and these two triangles form a "kite" quadrilateral. Hence, CD is perpendicular bisector of segment AE.



So, $\angle AGC = \angle CGE = \angle DGE = \angle AGD = 90\,°$

Now, on ΔFGC, $\angle CFG = 80°, \angle CGF = 90°$

So, $\angle FCG = \angle BCD = \angle x = 180° - 90° - 80° = 10°$